Problem Set 4 – Solutions

Problem 1

If $dX = \sigma dz$ then

$$\int_0^T dX = \int_0^T \sigma dz$$

$$X_T - X_0 = \sigma(z_T - z_0) = \sigma z_T$$

$$\Rightarrow E X_T = E(X_0 + \sigma z_T) = X_0 + \sigma E(z_T)$$

but $E(z_T) = z_0 = 0$

so that $E X_T = X_0$, thus X is a martingale.

Problem 2

$$\frac{dS_1}{S_1} = \mu_1\, dt + 0.4\, dz_1 + 0.3\, dz_2 - 0.2\, dz_3$$

$$\frac{dS_2}{S_2} = \mu_2\, dt + 0.5\, dz_1 - 0.2\, dz_2 - 0.1\, dz_3$$

a. $$\mathrm{Cov}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = \frac{1}{dt}\,\frac{dS_1}{S_1}\,\frac{dS_2}{S_2}$$

$$= 0.4 \times 0.5 - 0.3 \times 0.2 + 0.2 \times 0.1 = \frac{4}{25}$$

$$\mathrm{Var}\left(\frac{dS_1}{S_1}\right) = \frac{1}{dt}\left(\frac{dS_1}{S_1}\right)^2$$

$$= (0.4)^2 + (0.3)^2 + (-0.2)^2 = \frac{29}{100}$$

$$\mathrm{Var}\left(\frac{dS_2}{S_2}\right) = \frac{1}{dt}\left(\frac{dS_2}{S_2}\right)^2 = (0.5)^2 + (-0.2)^2 + (-0.1)^2 = \frac{3}{10}$$

$$\mathrm{Corr}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = \frac{4/25}{\sqrt{29/100}\,\sqrt{3/10}} \simeq 0.542.$$

b. We try first to find

$$z_4^* = z_1 + b_2 z_2 + b_3 z_3 \quad \text{so that}$$

$$(dz_4^*)\left(\frac{dS_1}{S_1}\right) = 0 \quad \text{and} \quad (dz_4^*)\left(\frac{dS_2}{S_2}\right) = 0$$

Thus,

$$\begin{cases} 1 \times 0.4 + b_2 \times 0.3 + b_3 \times (-0.2) = 0 \\ 1 \times 0.5 + b_2 (-0.2) + b_3 \times (-0.1) = 0 \end{cases}$$

$$\begin{cases} 0.3\, b_2 - 0.2\, b_3 = -0.4 \\ -0.2\, b_2 - 0.1\, b_3 = -0.5 \end{cases}$$

$$\big(0.3 - 2(-0.2)\big)\, b_2 = -0.4 - 2(-0.5)$$

$$0.7\, b_2 = 0.6$$

$$b_2 = 6/7$$

$$0.1\, b_3 = 0.5 - 0.2\, b_2 = 0.5 - 0.2 \times 6/7$$

$$b_3 = 23/7$$

Because $1^2 + (6/7)^2 + (23/7)^2 = \frac{614}{49} \approx 12.53$

$$a_1 = \frac{1}{\sqrt{12.53}} = 0.2825$$

$$a_2 = \frac{6/7}{\sqrt{12.53}} = 0.2421$$

$$a_3 = \frac{23/7}{\sqrt{12.53}} = 0.9282.$$

Problem 3

$$\frac{d\Lambda}{\Lambda} = -0.05\,dt - 0.3\,dz_1 - 0.5\,dz_2$$

$$\frac{dS_1}{S_1} = \mu_1\,dt + 0.5\,dz_1 + 0.2\,dz_2$$

$$\frac{dS_2}{S_2} = \mu_2\,dt - 0.1\,dz_1 + 0.4\,dz_2$$

a. $$\mathrm{Cov}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = \frac{1}{dt}\left(\frac{dS_1}{S_1}\right)\left(\frac{dS_2}{S_2}\right)$$

$$= 0.5 \times (-0.1) + 0.2 \times 0.4 = 0.03$$

$$\mathrm{Var}\left(\frac{dS_1}{S_1}\right) = \frac{1}{dt}\left(\frac{dS_1}{S_1}\right)^2 = 0.5^2 + 0.2^2 = 0.29$$

$$\mathrm{Var}\left(\frac{dS_2}{S_2}\right) = \frac{1}{dt}\left(\frac{dS_2}{S_2}\right)^2 = (-0.1)^2 + (0.4)^2 = 0.17$$

$$\mathrm{Corr}\left(\frac{dS_1}{S_1}, \frac{dS_2}{S_2}\right) = \frac{0.03}{\sqrt{0.29}\,\sqrt{0.17}} = 0.135$$

b. $$\mu_1 = r - \frac{1}{dt}\frac{d\Lambda}{\Lambda}\frac{dS_1}{S_1} = 0.05 + 0.3 \times 0.5 + 0.5 \times 0.2$$

$$= 0.30$$

$$\mu_2 = r - \frac{1}{dt}\frac{d\Lambda}{\Lambda}\frac{dS_2}{S_2} = 0.05 + 0.3 \times (-0.1) + 0.5 \times 0.4$$

$$= 0.22$$

Problem 4

$$d_1 = \frac{\ln(210/215) + (0.06 + \frac{1}{2}\, 0.45^2) \times 2}{0.45\sqrt{2}} = 0.4698$$

$$d_2 = d_1 - 0.45\sqrt{2} = -0.1667$$

$\phi(d_1) = 0.6807 \qquad \phi(d_2) = 0.4338$

Each contract is over 100 shares.

a. The trader needs to buy $100 \times 0.6807 = 68$ shares of the stock.

b. The trader needs to sell $100 \times 0.4338 \cong 43$ zero-coupon bonds expiring in 2 years with face value \$215.

c. The implicit leverage is given by

$$\frac{S\, C_S}{C} = \frac{210 \times 0.6807}{210 \times 0.6807 - 215\, e^{-0.06 \times 2} \times 0.4338}$$

$$= 2.37.$$

Problem 5

There are several ways to determine the price of the option written on an asset with no volatility.

1. If $\sigma = 0$, then the asset's dynamics are given by

$$\frac{dS}{S} = r\, dt$$

implying $S_T = S_0 e^{rT}$. We know for certain that the stock price in 6 months will be $S_T = 40\, e^{0.06 \times 6/12} = 41.22$.
Because the call will expire out of the money, it will pay zero and thus its price today must be zero.

2. Using the Black-Scholes model,

$$d_1 = \frac{\ln(S/K) + (r + \tfrac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}$$

$$= \frac{\ln(S/K) + rT}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

$$= -\frac{0.027}{\sigma} + 0.354\,\sigma \xrightarrow[\sigma \to 0]{} -\infty$$

Thus $\phi(d_1) \xrightarrow[\sigma \to 0]{} 0$

Since $d_2 = d_1 - \sigma\sqrt{T}$, we also have that

$$d_2 \xrightarrow[\sigma \to 0]{} -\infty \quad \text{so that} \quad \phi(d_2) \xrightarrow[\sigma \to 0]{} 0.$$

Therefore, $c = S\,\phi(d_1) - Ke^{-rT}\phi(d_2) = 0.$